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CONSEQUENCES OF DIRAC'S POSITRON THEORY

W. Heisenberg and H. Euler,
Leipzig, Germany

ABSTRACT: As a consequence of Dirac's positron theory one must modify Maxwell's equations for the vacuum since any electromagnetic field can produce pairs. These modifications are computed for the case when physical electrons and positrons are absent and when the field does not change appreciably over one Compton wavelength. The Lagrangian for the field is:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{c. c.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{c. c.}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2)\right\}.$$

$$\left(\begin{array}{l}\mathfrak{E}, \mathfrak{B} \text{ force on the electron} \\ |\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{\text{„}137\text{“}}\frac{e}{(e^2/mc^2)^2} = \text{"critical field strength."}\end{array}\right)$$

For fields that are small compared to $|\mathfrak{E}_k|$, the terms in the expansion of the Lagrangian describe scattering processes of light by light. The simplest term is already known from perturbation theory. Field equations for large fields are derived and are quite different from Maxwell's equations. The field equations are compared with those suggested by Born.

The fact that matter can be converted into radiation and radiation into matter leads to some fundamentally new aspects of quantum electrodynamics. /714*
One of the most important consequences of this interchangeability is that, even for processes in empty space, Maxwell's equations must be replaced by more complicated equations. Since fields of sufficient energy can generate matter, it will not generally be possible to separate processes in empty space from processes including matter. On the other hand, even with insufficient energy for generation of matter, the virtual possibility of materialization produces a "vacuum polarization" which again requires a change of Maxwell's equations. We will now study this vacuum polarization. As usual it will cause $\mathfrak{B}$ and $\mathfrak{E}$ to be different

*Numbers in the margin indicate pagination in the foreign text.

from $\mathfrak{D}$ and $\mathfrak{H}$, respectively, so that we can write

$$\left.\begin{aligned}\mathfrak{D} &= \mathfrak{E} + 4\pi\mathfrak{P},\\ \mathfrak{H} &= \mathfrak{B} - 4\pi\mathfrak{M},\end{aligned}\right\} \tag{1}$$

The polarizations $\mathfrak{P}$ and $\mathfrak{M}$ may be any complicated function of the fields and their derivatives at the point considered as well as the field in the vicinity of the point. For weak fields (the field must be small compared to e^2/hc times the field at the "edge of the electron," as will be shown later) $\mathfrak{P}$ and $\mathfrak{M}$ can be approximated by linear functions in $\mathfrak{E}$ and $\mathfrak{B}$. Uehling [1] and Serber [2] have computed the modifications of Maxwell's equations for this approximation. Another case of interest occurs when the field is not weak, but slowly varying, i.e. nearly constant over distances of the order of h/mc. Then $\mathfrak{P}$ and $\mathfrak{M}$ are functions of $\mathfrak{E}$ and $\mathfrak{B}$ at the point considered; in this approximation derivatives of $\mathfrak{E}$ and $\mathfrak{B}$ do not occur. We will later demonstrate that the expansion of $\mathfrak{P}$ and $\mathfrak{M}$ contains only odd powers of $\mathfrak{E}$ and $\mathfrak{B}$. The third-order terms obviously describe the scattering of light by light; they are already known [3]. The aim of this paper is a complete determination of the functions $\mathfrak{P}(\mathfrak{E}, \mathfrak{B})$ and $\mathfrak{M}(\mathfrak{E}, \mathfrak{B})$ for the case of slowly varying fields. For this purpose it is sufficient to compute the energy density of the field $U(\mathfrak{E}, \mathfrak{B})$ as a function of $\mathfrak{E}$ and $\mathfrak{B}$. The Hamiltonian formalism allows the derivation of wave equations from the energy density: one introduces the Lagrangian $\mathfrak{L}(\mathfrak{E}, \mathfrak{B})$ and defines /715

$$\mathfrak{D}_i = \frac{\partial\mathfrak{L}}{\partial\mathfrak{E}_i}, \quad \mathfrak{H}_i = -\frac{\partial\mathfrak{L}}{\partial\mathfrak{B}_i}, \tag{2}$$

$$U(\mathfrak{E}, \mathfrak{B}) = \frac{1}{4\pi}\Big[\sum_i \mathfrak{D}_i\mathfrak{E}_i - \mathfrak{L}\Big] = \frac{1}{4\pi}\Big(\sum_i \mathfrak{E}_i\frac{\partial\mathfrak{L}}{\partial\mathfrak{E}_i} - \mathfrak{L}\Big). \tag{3}$$

The Lagrangian is found from (3); $\underline{D}$ and $\underline{H}$ are determined by (2). The Lagrangian must be relativistically invariant and must therefore be a function of the invariants $\underline{E}^2-\underline{B}^2$ and $(\underline{EB})^2$ [3]. The calculation of $U(\underline{E}, \underline{B})$ is correlation with that of the energy-density of a matter-field which is coupled to the constant fields $\underline{E}$ and $\underline{B}$. Before considering this problem, we will briefly outline the mathematical framework of the positron theory [4] in order to correct some errors in the equations given previously. /716

1. Mathmetical Framework of the Positron Theory

The positron theory starts with Dirac's "density matrix" which is given by

$$(x'\,t'\,k'\,|R|\,x''\,t''\,k'') = \sum_{\substack{(n)\\ \text{occupied}\\ \text{states}}} \psi_n^*(x''\,t''\,k'')\,\psi_n(x'\,t'\,k'), \tag{4}$$

in the wave representation, and by

$$(x'\,t'\,k'\,|R|\,x''\,t''\,k'') = \psi^*(x''\,t''\,k'')\;\psi(x'\,t'\,k') \tag{5}$$

in the quantum theory of wave fields*. Another important matrix is the matrix R_s which is defined by

$$(x'\,t'\,k'\,|R_S|\,x''\,t''\,k'') = \tfrac{1}{2}\Big(\underbrace{\sum\nolimits_{(n)}}_{\text{occupied states}} - \underbrace{\sum\nolimits_{(n)}}_{\text{unoccupied states}}\Big)\,\psi_n^*(x''\,t''\,k'')\cdot\psi_n(x'\,t'\,k') \tag{6}$$

or

$$(x'\,t'\,k'\,|R_S|\,x''\,t''\,k'') = \tfrac{1}{2}[\psi^*(x''\,t''\,k'')\,\psi(x'\,t'\,k') - \psi(x'\,t'\,k')\,\psi^*(x''\,t''\,k'')] \tag{7}$$

The matrix R_s is a function of the differences $x'_\lambda - x''_\lambda = x_\lambda$ and $t' - t'' = t$; it becomes singular on the light-cone. With

$$c\,t = x_0 = -x^0;\quad x_i = x^i;\quad \xi_\lambda = \frac{x'_\lambda + x''_\lambda}{2}, \tag{8}$$

$A_0 = -A^0$; $A_i = A^i$ for the potentials and $\alpha^0 = -\alpha_0 = 1$ and $\alpha^i = \alpha_i$ for the Dirac matrices, we obtain

$$(x'\,k'\,|R_S|\,x''\,k'') = u\,\frac{\alpha^0\,x_0}{(x^\lambda\,x_\lambda)^2} - \frac{v}{x^\lambda\,x_\lambda} + w\log|x^\lambda\,x_\lambda|, \tag{9}$$

where**

$$u = -\frac{i}{2\pi^2}\,e^{+\frac{ei}{\hbar c}\int_{P'}^{P''} A^\lambda \mathrm{d}x_\lambda} \tag{10}$$

---

*Error in I: The primed and double-primed quantities on the right side are interchanged in I.
**Error in I: The exponent has a negative sign in I.

(The summation over identical Latin indices extends from 1 to 3, that for Greek indices from 0 to 3). The integration along a straight line extends from P' to P''.

The density matrix r characterizes the influence of matter. It is obtained from R_S with /717

$$r = R_S - S, \tag{11}$$

where S is defined by

$$S = e^{+\frac{e i}{\hbar c}\int_{P'}^{P''} A^\lambda d x_\lambda} \cdot S_0 + \frac{\bar{a}}{x_\lambda x^\lambda} + \bar{b} \log \left| \frac{x_\lambda x^\lambda}{C} \right| \tag{12}$$

S_0 is equal to the matrix R_S in space free of fields and matter. $\bar{a}, \bar{b}$ and C are defined by the following equations*:

$$\left.\begin{aligned} \bar{a} &= u \left\{ \frac{e i}{24 \hbar c} x_\varrho x^\sigma \alpha^\lambda \left(\frac{\partial F_{\lambda\sigma}}{\partial \xi_\varrho} - \delta_\lambda^\varrho \frac{\partial F_{\tau\sigma}}{\partial \xi_\tau} \right) - \frac{e^2}{48 \hbar^2 c^2} x_\varrho x_\sigma x^\tau \alpha^\varrho F^{\mu\sigma} F_{\mu\tau} \right\}, \\ \bar{b} &= u \left\{ \frac{e i}{24 \hbar c} \alpha^\lambda \frac{\partial F_{\tau\lambda}}{\partial \xi_\tau} + \frac{e^2}{24 \hbar^2 c^2} x_\lambda \alpha^\mu \left(F_{\tau\mu} F^{\tau\lambda} - \frac{1}{4} \delta_\mu^\lambda F_{\tau\sigma} F^{\tau\sigma} \right) \right\}, \\ C &= 4 \left(\frac{\hbar}{\gamma m c} \right)^2, \end{aligned}\right\} \tag{13}$$

γ is Euler's constant, $\gamma = 1{,}781\ldots$

The four-vector and the energy-momentum tensor are determined by r:

$$\left.\begin{aligned} s_\lambda(\xi) &= -e \sum_{k' k''} \alpha^\lambda_{k'' k'} (\xi k' | r | \xi k''), \\ U^\mu_\nu(\xi) &= \lim_{x \to 0} \left\{ i c \hbar \frac{\partial}{\partial x_\mu} - \frac{e}{2} \left[A^\mu\left(\xi + \frac{x}{2}\right) + A^\mu\left(\xi - \frac{x}{2}\right) \right] \right\} \\ &\qquad \sum_{k' k''} \alpha^\nu_{k'' k'} \left(\xi + \frac{x}{2}, k' | r | \xi - \frac{x}{2}, k'' \right). \end{aligned}\right\} \tag{14}$$

In the quantum theory of wave fields it is of advantage to expand the wave function in an orthogonal set of functions:

$$\psi(x, k) = \sum_n a_n u_n(x, k). \tag{15}$$

*A calculational error in equation (38) of I led to a different value of C. Contrary to common usage, in I the letter γ denoted the logarithm of Euler's constant.

The operators a_n can be rewritten as

$$a_n^* = N_n \Delta_n V_n; \quad a_n = V_n \Delta_n N_n, \tag{16}$$

where Δ_n converts the number N_n into $1-N_n$, and where $V_n = \prod_{t \leqq n} (1 - 2N_t)$.

We set

$$a_n' = a_n^* = -V_n \Delta_n N_n'; \quad a_n'^* = -N_n' \Delta_n V_n; \quad N_n' = 1 - N_n.$$

The Hamiltonian of the total system, in the new variables, is /718

$$\begin{aligned} H = \lim_{x \to 0} \int d\xi \Big\{ & -\Big(c i \hbar \frac{\partial}{\partial x_l} - \frac{e}{2}\Big[A^l\Big(\xi + \frac{x}{2}\Big) + A^l\Big(\xi - \frac{x}{2}\Big)\Big]\Big) \\ & \sum_{k' k''} \alpha^l_{k'' k'} \sum_{m,n} \frac{1}{2} (a_n^* a_m - a_m a_n^*) u_n^*\Big(\xi - \frac{x}{2}, k''\Big) u_m\Big(\xi + \frac{x}{2}, k'\Big) \\ & + \sum_{k' k''} \beta_{k'' k'} m c^2 \sum_{m,n} \frac{1}{2} (a_n^* a_m - a_m a_n^*) u_n^*\Big(\xi - \frac{x}{2}, k''\Big) u_m\Big(\xi + \frac{x}{2}, k'\Big) \\ & - \sum_{k'} \Big(c i \hbar \frac{\partial}{\partial x_0} - \frac{e}{2}\Big[A^0\Big(\xi + \frac{x}{2}\Big) + A^0\Big(\xi - \frac{x}{2}\Big)\Big]\Big)\Big(\xi + \frac{x}{2}, k' \,|S|\, \xi - \frac{x}{2}, k'\Big) \\ & + \frac{1}{8\pi}(\mathfrak{E}^2 + \mathfrak{B}^2)\Big\}. \end{aligned} \tag{17}$$

The coefficients of an expansion in powers of the elementary charge are:

$$\begin{aligned} H_0 &= \sum_{E_n > 0} N_n E_n - \sum_{E_n < 0} N_n' E_n + \sum_{\mathfrak{g} e} M_{\mathfrak{g} e} h \nu_{\mathfrak{g} e}, \\ H_1 &= \int d\xi \, e A^l(\xi) \sum_{k'' k'} \alpha^l_{k'' k'} \Big[\sum_{E_n > 0} N_n u_n^*(\xi k'') u_n(\xi, k') \\ & \quad - \sum_{E_n < 0} N_n' u_n^*(\xi k'') u_n(\xi, k') + \tfrac{1}{2} \sum_{n \neq m} (a_n^* a_m - a_m a_n^*) u_n^*(\xi, k'') u_m(\xi, k')\Big], \\ H_2 &= \int d\xi \Big[i \frac{e^2}{2\hbar c}\Big(\int A^\lambda d x_\lambda\Big)^2 \sum_{k'} \frac{\partial}{\partial x_0}\Big(\xi + \frac{x}{2}, k' \,|S_0|\, \xi - \frac{x}{2}, k'\Big) \\ & \quad + \frac{1}{12\pi^2} \frac{e^2}{\hbar c} \frac{x_\lambda x^\sigma}{x_\varrho x^\varrho} A^\lambda \Big(\frac{\partial F_{0\sigma}}{\partial \xi_0} - \frac{\partial F_{\tau\sigma}}{\partial \xi_\tau}\Big) + \frac{1}{24\pi^2} \frac{e^2}{\hbar c} \frac{x_\sigma x^\tau}{x_\varrho x^\varrho} F^{\mu\sigma} F_{\mu\tau} \\ & \quad - \frac{1}{12\pi^2} \frac{e^2}{\hbar c} \log \Big|\frac{x_\varrho x^\varrho}{C}\Big| \Big(F^{\tau 0} F_{\tau 0} - \frac{1}{4} F_{\tau\mu} F^{\tau\mu}\Big)\Big], \\ H_3 &= -\frac{1}{6} \frac{e^3}{\hbar^2 c^2} \int d\xi \, (A^\lambda x_\lambda)^3 \frac{\partial}{\partial x_0} \sum_{k'} \Big(\xi + \frac{x}{2}, k' \,|S_0|\, \xi - \frac{x}{2}, k'\Big), \\ H_4 &= -\frac{1}{12\pi^2} \Big(\frac{e^2}{\hbar c}\Big)^2 \frac{1}{\hbar c} \int d\xi \frac{(A^\lambda x_\lambda)^4}{(x_\varrho x^\varrho)^2}. \end{aligned} \tag{18}$$

2. Calculation of the Energy Density in the Wave Representation

The Lagrangian for the modified Maxwell's equations must be a function of only the invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$. It is therefore sufficient to find an energy-density of the matter-field as a function of two independent field components. For example, it is sufficient to determine the energy-density of matter in a constant electric and parallel constant magnetic field. In these constant fields one must examine the state of the matter-field which corresponds to the absence of matter. /719
This state is obviously that with the lowest energy. The lowest energy in the wave representation [Eqs. (4) and (6)] is given when all negative-energy electron levels are occupied and all positive-energy electron levels are empty. In the presence of a magnetic field, stationary electron states can again be divided into those with negative and those with positive energy. The lowest-energy state of the matter-field can thus be found in the same way with a magnetic field or without any fields.

The situation changes when an electron field is present. The potential energy increases linearly with a coordinate. Any value of the energy between $-\infty$ and $+\infty$ is possible. Eigenfunctions for different eigenvalues can be made identical by a mere shift in position. An unambiguous classification into positive and negative eigenvalues is impossible.

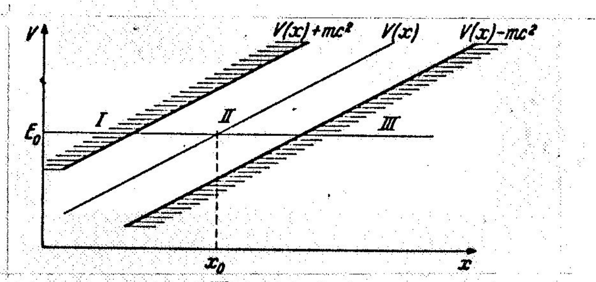

Figure 1.

The physical reason for this difficulty is related to the fact that electron-positron pairs can be generated spontaneously in a constant electric field. A complete calculation of this problem was given by Sauter [5]. Figure 1 shows the potential energy V(x), $V(x) + mc^2$ and $V(x) - mc^2$ as a function of the coordinate for an electron field parallel to the x-axis. Sauter's calculations show that eigenfunctions to an eigenvalue E_0 are large in regions I and III, and that they decrease exponentially within region II. This means that a wave-function which is large in one region, e. g. region I, will gradually taper off in region III. According to Sauter, the transmission coefficient of region II, which is equivalent here to a potential wall (German: Gamowberg, Gamow's mountain), is of order of magnitude $e^{-\frac{m^2c^3}{\hbar e|\mathfrak{E}|}\pi}$. This can be rewritten as $e^{-\frac{|\mathfrak{E}_k|}{|\mathfrak{E}|}\pi}$, where $|\mathfrak{E}_k| = \frac{m^2c^3}{\hbar c}$ is the critical field strength. Pair-creation occurs infrequently and can be neglected as long as /720
$|\mathfrak{E}| \ll |\mathfrak{E}_k|$. In this case it should be possible to find solutions of the Dirac equation which replace the eigenfunctions. These solutions will, for example, be large in region I and small (of order $e^{-\frac{|\mathfrak{E}_k|}{|\mathfrak{E}|}\frac{\pi}{2}}$) at every place in region III, at least for some time. Other solutions will be large in III and will almost completely disappear in I. With this part of the calculation accomplished, the lowest-energy state can be found: all electron levels whose eigenfunctions are large in III only must be occupied; all other levels must be unoccupied. The energy of such an

electron state is the difference between the energy density at the point x_o and E_o [compared equation (31)]. If the electric field is turned off adiabatically, then this state of the system goes into the state of field-free space where only electron levels with negative energy are occupied.

Our mathematical analysis follows Sauter's investigation. The Dirac equation in the presence of an external magnetic field $\mathfrak{B}$ and an electric field $\mathfrak{E}$ (both in direction of the x-axis) can be written as:

$$\left\{\frac{i\hbar}{c}\frac{\partial}{\partial t} - \frac{e}{c}|\mathfrak{E}|\,x + \alpha_1 i\hbar\frac{\partial}{\partial x} + \alpha_2 i\hbar\frac{\partial}{\partial y} + \alpha_3\left(i\hbar\frac{\partial}{\partial z} - \frac{e}{c}|\mathfrak{B}|\,y\right) - \beta mc\right\}\psi = 0. \tag{19}$$

The motion along the y- and z- axis can be separated from that in direction x:

$$\psi = e^{\frac{i}{\hbar}(p_z z - Et)} \cdot u_n(y) \cdot \chi. \tag{20}$$

We define an operator K by the relation

$$K = +\alpha_2 i\hbar\frac{\partial}{\partial y} + \alpha_3\left(-p_z - \frac{e}{c}|\mathfrak{B}|\,y\right) - \beta mc. \tag{21}$$

We therefore obtain:

$$K^2\psi = \left\{-\hbar^2\frac{\partial^2}{\partial y^2} - i\hbar\,\alpha_2\alpha_3\frac{e|\mathfrak{B}|}{c} + \left(p_z + \frac{e}{c}|\mathfrak{B}|\,y\right)^2 + m^2c^2\right\}\psi. \tag{22}$$

This equation can be interpreted as wave equation for the function $u_n(y)$ which is as yet undetermined. With

$$y = \eta\sqrt{\frac{\hbar c}{e|\mathfrak{B}|}} - \frac{c\,p_z}{e|\mathfrak{B}|}, \tag{23a}$$

we have

$$u_n(y) = H_n(\eta)\,e^{-\frac{\eta^2}{2}}\left(2^n \cdot n! \cdot \sqrt{\pi}\right)^{-1/2}\left(\frac{\hbar c}{e|\mathfrak{B}|}\right)^{-1/4} \tag{23}$$ /721

since (22) is essentially the Schrödinger equation for the harmonic oscillator. $H_n(y)$ is the nth Hermite polynomial. From (23) one obtains

$$K^2 u_n \chi = \left\{m^2 c^2 + \frac{e|\mathfrak{B}|\hbar}{c}(2n+1+\sigma_x)\right\} u_n \chi \qquad (n = 0, 1, 2, \cdots) \tag{24}$$

The operators K and α_1 anticommute. The wave equation (19) can be written in the form

$$\left\{\frac{E - e|\mathfrak{E}|x}{c} + \alpha_1 i\hbar \frac{\partial}{\partial x} + K\right\}\psi = 0 . \tag{25}$$

A canonical transformation can be applied to χ so that the matrix σ_x is diagonalized and K and α_1 become

$$K = \sqrt{m^2c^2 + \frac{e|\mathfrak{B}|\hbar}{c}(2n+1+\sigma_x)} \cdot \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}; \quad \alpha_1 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} . \tag{26}$$

The two matrices are based on another index which is independent of the spin-orientation (i. e. the "ϱ"-coordinate). σ_x can be considered to be a simple number $(\sigma_x = \pm 1)$; with the abbreviations

$$\left.\begin{aligned} \xi &= \sqrt{\frac{1}{\hbar c e|\mathfrak{E}|}}\,(e|\mathfrak{E}|x - E), \\ k &= \sqrt{\frac{c}{\hbar e|\mathfrak{E}|}\left(m^2c^2 + \frac{e|\mathfrak{B}|\hbar}{c}(2n+1+\sigma_x)\right)}. \end{aligned}\right\} \tag{27}$$

We obtain the equations

$$\left.\begin{aligned} \left(\frac{d}{d\xi} - i\xi\right)f + kg &= 0, \\ \left(\frac{d}{d\xi} + i\xi\right)g + kf &= 0, \end{aligned}\right\} \tag{28}$$

where f and g are the components of the function χ with respect to the "ϱ"-index. The form of the equations (28) is identical to that of Sauter's equation (12). The

difference lies in the meaning of the quantity k as well as in the fact that the system of equations (28) should be written twice: for $\sigma_z = +1$ and $\sigma_z = -1$. /722
Sauter obtains two sets of solutions for equation (28):

$$\left.\begin{aligned}
f_1 &= -\frac{1}{2\sqrt{\pi}}|\xi|\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-\frac{1}{2}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}} ds,\\
g_1 &= -\frac{1}{2\sqrt{\pi}}\frac{k|\xi|}{2\xi}\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-\frac{1}{2}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-1} ds,\\
f_2 &= -\frac{1}{2\sqrt{\pi}}\frac{k|\xi|}{2\xi}\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-1}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-\frac{1}{2}} ds,\\
g_2 &= -\frac{1}{2\sqrt{\pi}}|\xi|\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-\frac{1}{2}} ds.
\end{aligned}\right\} \qquad (29)$$

The path of integration begins at $+\infty$, passes the singular points at $+i/2$ and $-i/2$, and returns to $+\infty$.

As mentioned above, we will neglect pair-production in our calculations. The parts of the functions f and g, which vanish in one half-space, will therefore be accepted as eigenfunctions. For example, we set

$$f_1^1 = \begin{cases} f_1 & \text{for } \xi > 0, \\ 0 & \text{,, } \xi \leqq 0, \end{cases} \qquad f_1^2 = \begin{cases} 0 & \text{for } \xi > 0, \\ f_1 & \text{,, } \xi \leqq 0 \end{cases} \quad \text{etc.} \qquad (30)$$

The new functions f_1^1 etc. do not exactly correspond to stationary states. They represent wave-packets which have a very low probability for diffusion into the region that was initially empty. For the density matrix we consider the states f_1^1, g_1^1, etc. to be occupied and f_1^2, g_1^2, etc. to be unoccupied. We have doubled the number of "states" by the procedure (30). One thus obtains twice the density matrix if all f_1^1, g_1^1, f_2^1, g_2^1 are assumed to be occupied, and f_1^2, g_1^2, f_2^2, g_2^2 are assumed to be unoccupied.

For a calculation of the energy density in vacuum according to the method of Section 1, one would first calculate the density matrix for a finite distance between the points r' and r". Then one would subtract the singular S-matrix (given in Section 1), compute the energy density and finally go to the limit r' = r". It is more convenient to start the calculation immediately with r' = r". On the other hand, we will sum the stationary states up to finite energies, or, we will assure convergence of the summation by an additional factor $e^{-\text{const}\cdot[k^2-(mc^2)^2]}$, which amounts to about the same. The constant in the exponent is then set equal to zero; /723
some terms in the energy-density matrix will become singular, but they will be

compensated by the corresponding terms of the S-matrix. The remaining non-singular terms will yield the desired result.

Before writing the density matrix, the eigenfunctions must be normalized. One might consider the eigenfunctions to be limited to a large distance L in the directions of x and z (the eigenfunctions $U_n(y)$ are already normalized). For the z-direction, one obtains the normalization factor $1/\sqrt{L}$. Because of the asymptotic behavior of Sauter's eigenfunctions ([5], equation (22)) we get a factor $2\frac{1}{\sqrt{L}}e^{-\frac{k^2\pi}{4}}$ in direction x. A summation over all states must be carried out, over all momenta of the form

$$p_z = \frac{h}{L}\cdot m + \text{const}$$

and over all energies of the form $E = \frac{hc}{(L/2)}m + \text{const}$. The sums can be converted into integrals with the differential $\frac{\mathrm{d}p_z}{h}\cdot\frac{\mathrm{d}E}{2hc}$, where we have again omitted factors of $\frac{1}{\sqrt{L}}$ in the eigenfunctions. For a calculation of the density at a point x_0 the energy of the state is equal to the difference $E - e|\mathfrak{E}|\,x_0$. An expression for the energy density corresponding to the matrix R_s (compare Section 1) is then:

$$U = \frac{1}{2}\sum_0^{\infty}{}^{(n)}\sum_{-1}^{+1}{}^{(\omega)}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}p_z}{h}\int\frac{\mathrm{d}E}{hc}(E - e|\mathfrak{E}|\,x)\,u_n^2(y)\,e^{-\frac{k^2\pi}{2}} \cdot \begin{bmatrix} |f_1^1|^2 + |g_1^1|^2 - |f_1^2|^2 - |g_1^2|^2 \\ + |f_2^1|^2 + |g_2^1|^2 - |f_2^2|^2 - |g_2^2|^2 \end{bmatrix} e^{-\alpha\left(\xi^2 - \frac{1}{\alpha}\right)}, \tag{31}$$

[(a will be defined in (33)]; with (23a) we obtain

$$U = -\sum_0^{\infty}{}^{(n)}\sum_{-1}^{+1}{}^{(\omega)}\frac{e|\mathfrak{B}|}{c}\,\frac{\hbar e|\mathfrak{E}|}{h^2}\int\limits_{-\infty}^{+\infty}\mathrm{d}\xi\,|\xi|\,e^{-\frac{k^2\pi}{2}+\frac{\alpha}{a}} \cdot \tfrac{1}{2}[|f_1|^2 + |g_1|^2 + |f_2|^2 + |g_2|^2]\,e^{-\alpha\xi^2}. \tag{32}$$

This expression will later be used to discuss the behavior for $\alpha \to 0$. We introduce the abbreviations

$$\frac{e|\mathfrak{E}|\,\hbar}{m^2c^3} = a; \qquad \frac{e|\mathfrak{B}|\,\hbar}{m^2c^3} = b. \tag{33}$$

a and b are dimensional-less and express the ratios of the field strengths to the critical field strength $|\mathfrak{E}_k|$, i. e. to "1/137 of the field strength at the boundary of the electron." /724

Insertion of (29) into (33) yields

$$U = -\frac{1}{2}\sum_{0}^{\infty}{}^{(n)}\sum_{-1}^{+1}{}^{(\sigma)} \cdot a \cdot b \cdot mc^2\left(\frac{mc}{\hbar}\right)^3 \cdot \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} d\xi\,|\xi|\,e^{-\frac{k^2\pi}{2}+\frac{\alpha}{a}-\alpha\xi^2} \int ds_1 \int ds_2\, e^{-\xi^2(s_1+s_2)}\, e^{-\frac{k^2}{4i}\log\frac{\left(s_1+\frac{i}{2}\right)\left(s_2+\frac{i}{2}\right)}{\left(s_1-\frac{i}{2}\right)\left(s_2-\frac{i}{2}\right)}} \cdot h(s_1 s_2), \tag{34}$$

where $h(s_1\ s_2)$ is equal to

$$h(s_1 s_2) = \frac{1}{2\pi\left(s_1+\frac{i}{2}\right)^{1/2}\left(s_2-\frac{i}{2}\right)^{1/2}}\left[\xi^2 + \frac{k^2}{4\left(s_1-\frac{i}{2}\right)\left(s_2+\frac{i}{2}\right)}\right].$$

The integration over ξ yields

$$U = -\frac{1}{2}\sum_{0}^{\infty}{}^{(n)}\sum_{-1}^{+1}{}^{(\sigma)} a \cdot b \cdot mc^2\left(\frac{mc}{\hbar}\right)^3 \cdot \int ds_1 \int ds_2\, e^{-\frac{k^2\pi}{2}+\frac{\alpha}{a}-\frac{k^2}{4i}\log\frac{\left(s_1+\frac{i}{2}\right)\left(s_2+\frac{i}{2}\right)}{\left(s_1-\frac{i}{2}\right)\left(s_2-\frac{i}{2}\right)}} \cdot \frac{1}{8\pi^3}\left(s_1+\frac{i}{2}\right)^{-1/2}\left(s_2-\frac{i}{2}\right)^{-1/2}\left[\frac{1}{(s_1+s_2+\alpha)^2} + \frac{k^2}{4(s_1+s_2+\alpha)\left(s_1-\frac{i}{2}\right)\left(s_2+\frac{i}{2}\right)}\right]. \tag{35}$$

The first of the two integrations (e.g. that over s_1) is fairly easy; the path of integration is deformed so that only one loop remains around the singularity $s_1 = -s_2 - \alpha$. If S_2 in the result is replaced by $s = s_2 + \alpha/2$, then one obtains

$$U = \sum_{0}^{\infty}{}^{(n)}\sum_{-1}^{+1}{}^{(\sigma)} a \cdot b \cdot mc^2\left(\frac{mc}{\hbar}\right)^3 \cdot f(k) \cdot e^{\frac{\alpha}{a}}, \tag{36}$$

where

$$f(k) = -\int \mathrm{d}s \frac{1}{32\pi^2}\left(s-\frac{i}{2}+\frac{\alpha}{2}\right)^{-\frac{3}{2}}\left(s-\frac{i}{2}-\frac{\alpha}{2}\right)^{-\frac{1}{2}}\left(s+\frac{i}{2}+\frac{\alpha}{2}\right)^{-1}\left(s+\frac{i}{2}-\frac{\alpha}{2}\right)^{-1}$$

$$\left[k^2(i-\alpha)+2\left(s+\frac{i}{2}\right)^2-\frac{\alpha^2}{2}\right] e^{-\frac{k^2\pi}{2}-\frac{k^2}{4i}\log\frac{s^2-\left(\frac{i-\alpha}{2}\right)^2}{s^2-\left(\frac{i+\alpha}{2}\right)^2}}. \quad (37)$$

The path of integration is shown in Fig. 2; it goes from + ∞ through the four /725
singularities of the term under the integral ($s = \pm i/2 \pm \alpha/2$) and back to + ∞. One may also integrate along the imaginary axis from + ∞ to - ∞. The principle contribution comes from the part of the path which lies between the singularities. There one can develop the logarithm in the exponent into powers of α:

$$\log\frac{s^2-\left(\frac{i-\alpha}{2}\right)^2}{s^2-\left(\frac{i+\alpha}{2}\right)^2} = -2\pi i + \frac{i\alpha}{s^2+\frac{1}{4}} + \frac{i\alpha^3}{4(s^2+\frac{1}{4})^2} - \frac{i\alpha^3}{12(s^2+\frac{1}{4})^3} + \cdots \quad (38)$$

From now on we will assume that the electric field is small compared to the critical field strength $|\mathfrak{E}_k|$, i. e. $a \ll 1$, and hence

$$k^2 = \frac{1}{a}[1 + b(2n+1+\sigma_x)] \gg 1.$$

Then the expression $-\frac{k^2\alpha}{4s^2+1}$ in the exponent cannot be simplified. Higher-order terms in the exponent can, however, be considered to be small (we are interested in the limit $\lim \alpha \to 0$), and may be developed into a series. Thus we obtain an expression for f(k) which is of the form

$$f(k) = \frac{1}{2\pi^2}\int_{+i/2}^{-i/2} \mathrm{d}s\, e^{-\frac{k^2\alpha}{4s^2+1}}(1+4s^2)^{-2}[A + Bk^2 + Ck^4 \ldots]. \quad (39)$$

Before integrating, it is of advantage to sum over n and σ according to

$$\sum_{-1}^{+1}{}^{(\sigma)} \sum_{0}^{\infty}{}^{(n)} g\left(n+\frac{1+\sigma_x}{2}\right) = \left(\sum_0^\infty + \sum_1^\infty\right) g(n)$$

$$= \lim_{n' \to \infty}\left\{g(0) + 2\sum_1^{n'} g(n) + 2\int_{n'+1/2}^{\infty} g(n)\,dn + \tfrac{1}{2} g'(n'+\tfrac{1}{2}) + \cdots\right\}. \quad (40)$$

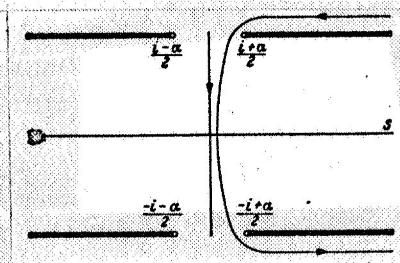

Figure 2.

It is apparent that higher-order terms of Euler's sum rule do not contribute to the final result. With $\alpha/a = \varepsilon$, ($\gamma = 1.781$ is Euler's constant) we obtain for $\varepsilon \to 0$: /726

$$\begin{aligned}
&\frac{U}{4\pi m c^2}\left(\frac{h}{mc}\right)^3 \\
&= \lim_{n'\to\infty}\Bigg\{-\frac{1}{\varepsilon^2}-\frac{1}{2\varepsilon}-\left(\frac{1}{8}+\frac{a^2}{12}+\frac{b^2}{12}\right)\log\frac{\gamma\varepsilon}{4}-\frac{1}{16}-\frac{b^2}{6}-\frac{a^2}{12} \\
&\qquad -\frac{1}{16}+\frac{b^2}{8}+\frac{[1+(2n'+1)b]^2}{16} \\
&\qquad -\frac{1}{8}[1+(2n'+1)b]^2\log[1+(2n'+1)b] \\
&\qquad +\frac{b^2}{24}\log[1+(2n'+1)b]+\frac{b}{2}\sum_1^{n'}(1+2nb)\log(1+2nb) \\
&\qquad +\frac{a^2}{12}\left\{b+2b\sum_1^{n'}\frac{1}{(1+2nb)}-\log[1+(2n'+1)b]\right\} \\
&\qquad +\frac{a^4}{80}\left\{b+2b\sum_1^{n'}\frac{1}{(1+2nb)^3}\right\} \\
&\qquad +\sum_{m=3}^{\infty}c_m a^{2m}\left\{b+2b\sum_1^{n'}\frac{1}{(1+2nb)^{2m-1}}\right\}\Bigg\}.
\end{aligned} \tag{41}$$

The coefficients c_m will be determined later.

The terms corresponding to the singular S-matrix must be subtracted from this result. The field-independent part of this singular energy density is easily obtained by repeating the calculation for plane waves

$$\begin{aligned}
U_S' &= -2\int c\sqrt{m^2c^2+p^2}\, e^{-\varepsilon\left(\frac{p}{mc}\right)^2}\frac{dp_x\,dp_y\,dp_z}{h^3} \\
&= 4\pi m c^2\left(\frac{mc}{h}\right)^3\left(-\frac{1}{\varepsilon^2}-\frac{1}{2\varepsilon}-\frac{1}{8}\log\frac{\gamma\varepsilon}{4}-\frac{1}{16}\right).
\end{aligned}$$

The calculation of the field-dependent parts of S is more difficult. According to equation (13), $\bar{a}$ and $\bar{b}$ contain the field strengths to the second power; the same

holds for U_s. The constant C in equation (13) is adjusted so that a vacuum polarization proportional to the field cannot occur for constant fields. Therefore, all terms of second order in the field must be subtracted, and only terms of higher order remain. If one expands in b for $b \ll 1$, then the whole first line at the right side of equation (41) must be subtracted. In order to verify this result we assumed r' and r'' of the density matrix to be different in x-direction $[\mathfrak{r}' - \mathfrak{r}'' = (x, 0, 0)]$ for the case where a = o, $b \neq 0$. Then we set $\alpha = \varepsilon = 0$ and subtracted the /727
the terms due to the S-matrix; the remainder was the part of U which has just been discussed. For the electric field, however, this calculation was too complicated. The total energy density is the sum of the classical Maxwell's energy density $\frac{1}{8\pi}(\mathfrak{E}^2 + \mathfrak{B}^2)$, and of Dirac's energy density $U - U_s$. With this and equation (3), one finds the Lagrangian

$$a \frac{\partial \mathfrak{L}}{\partial a} - \mathfrak{L} = 4\pi\left(U - U_s + \frac{1}{8\pi}(\mathfrak{E}^2 + \mathfrak{B}^2)\right).$$

We obtain

$$\begin{aligned}
\mathfrak{L} = {} & \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) \\
& + 16\pi^2 m c^2 \left(\frac{mc}{h}\right)^3 \lim_{n' \to \infty} \left\{ \frac{1}{16} - \frac{b^2}{8} - \frac{[1 + (2n'+1)b]^2}{16}\left(1 - 2\log[1 + (2n'+1)b]\right) \right. \\
& \qquad - \frac{b^2}{24}\log[1 + (2n'+1)b] - \frac{b}{2}\sum_1^{n'}(1 + 2nb)\log(1 + 2nb) \\
& \qquad + \frac{b a^2}{12}\left(1 + 2\sum_1^{n'}\frac{1}{1 + 2nb} - \frac{1}{b}\log[1 + (2n'+1)b]\right) \\
& \qquad + \frac{b a^4}{90}\left(1 + 2\sum_1^{n'}\frac{1}{(1 + 2nb)^3}\right) \\
& \qquad \left. + \sum_{m=3}^{\infty}\frac{c_m}{2m-1}a^{2m} b \left(1 + 2\sum_1^{n'}\frac{1}{(1 + 2nb)^{2m-1}}\right)\right\}.
\end{aligned} \tag{42}$$

For small magnetic fields one can apply Euler's summation rule again to equation (42) in order to find a power-series expansion in b. $\mathfrak{L}(a, b)$ is a function of the invariants $a^2 - b^2$ and $a^2 b^2$, so that from $\mathfrak{L}(a, b)$ follows $\mathfrak{L}(a, 0) = \mathfrak{L}(0, ia)$ —. As a consequence one can use this relation to calculate the unknown coefficients c_m which would otherwise be difficult to calculate. We computed c_2 and c_3 to show that the results of a direct calculation of c_m are the same; so far we have been unable to find a general proof for this statement. For small fields $(a \ll 1, b \ll 1)$: we obtain with this method

$$\mathfrak{L} \approx \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 16\pi^2 m c^2 \left(\frac{mc}{h}\right)^3 \left[\frac{(a^2 - b^2)^2 + 7(ab)^2}{180} + \frac{13(ab)^2(a^2 - b^2) + 2(a^2 - b^2)^3}{630} \cdots\right]. \tag{43}$$

For the other limit $(a \ll 1,\ b \gg 1)$ we have /728

$$\mathfrak{L} \approx \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 16\pi^2 m c^2 \left(\frac{mc}{h}\right)^3 \left\{ b^2 \left[\frac{1}{12}\log b - 0{,}191\right] + \frac{b}{4}[\log b - 0{,}145] + \frac{\log b}{8} + 0{,}202 - \frac{a^2}{12}[\log b + 0{,}116] + b\left[\frac{a^2}{12} + \frac{a^4}{90} + \cdots\right] + \cdots \right\} \tag{44}$$

In order to gain insight into the general behavior of $\mathfrak{L}$ for arbitrary fields, we attempted to find an integral representation of $\mathfrak{L}$. Starting from the usual integral representation of the zeta-function, we have

$$\begin{aligned}\mathfrak{L} &= \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 m c^2 \left(\frac{mc}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ -a\eta \operatorname{ctg} a\eta \cdot b\eta \operatorname{Ctg} b\eta + 1 + \frac{\eta^2}{3}(b^2 - a^2) \right\} \\ &= \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 m c^2 \left(\frac{mc}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \\ &\quad \left\{ -iab\eta^2 \frac{\operatorname{Cos}(b+ia)\eta + \operatorname{Cos}(b-ia)\eta}{\operatorname{Cos}(b+ia)\eta - \operatorname{Cos}(b-ia)\eta} + 1 + \frac{\eta^2}{3}(b^2 - a^2) \right\}.\end{aligned} \tag{45}$$

In the last expression for $\mathfrak{L}$ it is particularly obvious that $\mathfrak{L}$ depends only on the invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$. The cosine-terms can be expanded in terms of the square of the argument $(b+ia)^2 = b^2 - a^2 + 2i(ab)$ and $(b-ia)^2 = b^2 - a^2 - 2iab$. The overall expression is real; it can therefore be represented as a power-series in b^2-a^2 and $(ab)^2$. In general, the latter terms can be replaced by $\frac{\mathfrak{B}^2 - \mathfrak{E}^2}{|\mathfrak{E}_k|^2}$ and $\frac{(\mathfrak{E}\mathfrak{B})^2}{|\mathfrak{E}_k|^4}$, respectively. $\left(|\mathfrak{E}_k| = \frac{m^2 c^3}{e\hbar}\right)$. The Lagrangian for fields of arbitrary orientation becomes

$$\begin{aligned}\mathfrak{L} &= \tfrac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) \\ &+ \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B}) \frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) + \text{c.c.}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2 - \mathfrak{B}^2 + 2i(\mathfrak{E}\mathfrak{B})}\right) - \text{c.c.}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2 - \mathfrak{E}^2) \right\}. \quad (45\,\text{a})\end{aligned} \tag{45a}$$

The first term of the expansion in (43) agrees with the results of Euler and Kochel (loc. cit.).

The convergence of this power-series expansion must be investigated in more detail. For a = o the intergal (45) converges for any value of b. But for a $a \neq 0$ the integral becomes meaningless at $\eta = \pi/a,\ 2\pi/a,\ \cdots$, where ctg a η becomes infinite. The power-series expansion in a (which was used in the derivation) will therefore be semiconvergent. The intergal (45) can be made unambiguous if the path of integration is such that the singular points $\pi/a,\ 2\pi/a$ are by-passed. Then the integral (45) will contain additional imaginary terms which, at first sight, cannot be interpreted physically. The meaning becomes clear when the magnitude of these terms is estimated. The integral (45) has a value $-\frac{2\,i}{\pi}\cdot 4\,a^2\,mc^2\left(\frac{m\,c}{h}\right)^3 e^{-\frac{\pi}{a}}$ at the singular point $\eta = \pi/a$ (for $b = 0$). This is just the order of magnitude of the terms that describe pair-production in an electric field. The integral (45) appears to be similar to the perturbation-theory integration over a periodically vanishing denominator. One may assume that convergence of the integral is provided by a damping term which reflects the frequency of the periodic resonance process. The result of a calculation which bypasses the singular points will be correct up to terms which are of order of magnitude corresponding to the resonance frequency. According to (43) and (44), deviations from Maxwell's theory remain small as long as $\mathfrak{E}$ and $\mathfrak{B}$ are small compared to the electric field at distance $\sqrt{137}\cdot e^2/mc^2$ from the center of the electron. Even if the magnetic field is larger than this value, the corrections to Maxwell's equations will be small (of order $\frac{1}{3\pi}\frac{e^2}{\hbar c}$, compared to the original terms) as long as log b is of the order of 1. For example, deviations from the usual Coulomb-force between two protons, due to (43) and (44) will always remain small. On the other hand, one must consider that the additional terms for a Coulomb-field (which contain derivatives of the field strengths) may be more important than those included in equations (43) and (44). /729

3. Implications of the Result for the Quantum Theory of Wave Fields

The results of the derivation in the preceding section cannot be adapted immediately to the quantum theory of wave fields. It can easily be shown that the equations obtained above do not describe the state of matter in a homogeneous /730 field as seen by the quantum theory of wave fields. Consider first that the state of matter discussed in the previous section is the "unperturbed" state. Then there are matrix elements of the perturbation energy which describe the simultaneous production of a photon and an electron-positron pair. Even if the energy is insufficient to generate such particles, these matrix elements will give rise to a second-order perturbation energy. This is due to the virtual possibility of generation and annihilation of a photon and a pair; the appropriate calculation diverges. The appearance of perturbation terms becomes intuitively obvious if one considers, for example, that circular orbits in a magnetic field are not really stationary states — electrons in such states can radiate. The crucial point for the "physics" of the calculations in the preceding section is that this radiation need not be included in the classical theory of wave-fields. The solution obtained demonstrates that the charge and current density of matter vanishes and that it therefore does not radiate. This is in contrast with the quantum theory of waves in which a remainder of this radiation appears in form of a second-order divergent perturbation energy.

The same type of perturbation energy appears in field-free vacuum ("self-energy of the vacuum"). Such self-energies appear whenever one calculates the second- and higher-order energy contributions of virtual transitions into a different state and return to the initial state. So far self-energies have always been neglected. For example, the interaction cross section for Compton scattering is obtained by a perturbation calculation up to the second order. By including fourth-order terms, one would get contributions like those just mentioned; the result would not converge. The scattering of light by light (loc. cit.) is computed in a perturbation calculation up to the fourth order (this is the lowest order which contributes to the process under consideration). In this case, the sixth-order contributions would diverge. Such calculations have so far been successful (e.g. for the Klein-Nishina equation) which seems to indicate that the omission of the diverging higher-order contributions leads to correct results. If this is true, then the results of Section 2 can be transferred to the quantum theory of waves. /731
This is physically reasonable since the corresponding equivalent to the radiation terms could not be interpreted. Each term in the expansion of the energy density in powers of $\mathfrak{E}$ and $\mathfrak{B}$ can now be related to a scattering process; the cross section can be determined with the appropriate term. For example, fourth-order terms describe ordinary scattering of light by light. Sixth-order terms describe processes where three photons are scattered off each other, etc. Independent of whether the omission of higher-order terms is permissible from the point of view of physics, each term in the expansion of the preceding section must agree with the result of a direct calculation of the particular scattering process. This perturbation calculation, according to the quantum theory of waves, must be carried out up to the lowest-order term which contributes to the process considered. The underlying reason is that both methods neglect the contribution from terms which correspond to the creation and annihilation of a photon and a pair. (The accuracy of the calculation can be verified by comparing the fourth-order terms with the results of a direct calculation of the scattering of light by light [6]). It might therefore by possible to apply the solution for $|\mathfrak{B}| \geqq |\mathfrak{E}_k|$ to known results. This is certainly not possible for $|\mathfrak{E}| \gtrsim |\mathfrak{E}_k|$, since pair-production actually occurs in large electric fields; the calculations made above do not apply in this case.

4. Physical Consequences of the Result

The results of Section 2 are quite similar in form to the formulae which Born [7] used to modify Maxwell's equations. Instead of the classical Lagrangian $\mathfrak{E}^2 - \mathfrak{B}^2$, Born's calculation uses a complicated function of the invariants $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$. The order of magnitude of the first terms in the expansion agree with (43) due to the numerical value of e^2/hc (compare [6]). On the other hand, one must also emphasize the differences between the results. Born uses the modified Maxwell's equations as the basis of his theory while these modifications /732
appear as indirect consequences of the virtual possibility of pair-production in the Dirac theory. The fact that there are more modifications to Maxwell's equations than were calculated above (terms with higher-order derivatives of the field strength will be added [8]), is also related to this consequence of the Dirac theory. In particular, the question of the self-energy of electrons cannot by answered by exclusive consideration of these modifications. Born's theory showed that modifications of Maxwell's equations of the order of magnitude

considered here may be sufficient to eliminate difficulties that result from an infinite self-energy. This result provides an important clue for the further development of the theory.

In this connection, one must also ask whether results of the Dirac theory concerning the scattering of light by light, etc., can be considered to be final, or whether future theories can be expected to yield other results. The positron theory and the present state of quantum electrodynamics must undoubtedly be considered to be preliminary. In particular, the rules for the formation of the S-matrix (inhomogeneity of the Dirac equation) seem to be arbitrary. Therefore, future deviations from the present theory are possible in this respect. Such changes will have particular influence upon modifications of Maxwell's equations. The present theory can be assumed to provide the correct order of magnitude and qualitative form of these modifications. Of course, it is as yet impossible to predict the final form of Maxwell's equations in the future quantum field theory; it will be essential to consider the total of all processes involving high-energy particles (e.g. "shower" formation).

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